MA 106 : Linear Algebra

Tutorial 7

Soumya Chatterjee

IIT Bombay

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Question 7

Which quadric surface does the equation $7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy - 36 = 0$ describe? Explain by reducing the quadratic form involved to a diagonal form. Express x, y, z in terms of the new coordinates u, v, w.

- $\mathbf{A} = \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix}$
- $\det \begin{bmatrix} 7-\lambda & -1 & -10 \\ -1 & 7-\lambda & 10 \\ -10 & 10-\lambda & -2 \end{bmatrix} = (6-\lambda)\det \begin{bmatrix} 8-\lambda & 10 \\ 20 & -2-\lambda \end{bmatrix}$
- $= (6-\lambda)(\lambda^2 - 6\lambda - 216)$
- Eigen values: 6, 18 and -12
- 2 positive and 1 negative $\Rightarrow$ elliptical cone or 1-sheeted hyperboloid
- Eigen vectors: $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$, $\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^\top$, $\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^\top$

- Eigen vectors are already orthogonal. Normalize
- $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0\end{bmatrix}^\top$, $\frac{1}{\sqrt{3}}\begin{bmatrix}-1 & 1 & 1\end{bmatrix}^\top$, $\frac{1}{\sqrt{6}}\begin{bmatrix}1 & -1 & 2\end{bmatrix}^\top$
- $\mathbf{C} = \frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{3} & -\sqrt{2} & 1\\ \sqrt{3} & \sqrt{2} & -1\\ 0 & \sqrt{2} & 2\end{bmatrix}$
- $\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\\ \mathbf{z}\end{bmatrix} = \mathbf{C}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{w}\end{bmatrix}$
- $= \left[\frac{\sqrt{2}u}{2} - \frac{\sqrt{3}v}{3} + \frac{\sqrt{6}w}{6},\ \frac{\sqrt{2}u}{2} + \frac{\sqrt{3}v}{3} - \frac{\sqrt{6}w}{6},\ \frac{\sqrt{3}v}{3} + \frac{\sqrt{6}w}{3}\right]^\top$
- Substituting in $7x^2 - 2xy - 20xz + 7y^2 + 20yz - 2z^2 - 36 = 0$ and simplifying, we get, $\frac{u^2}{6} + \frac{v^2}{2} - \frac{w^2}{3} = 1$
- Therefore, it is a 1-sheeted hyperboloid

Question 8

Let Y be a subspace of $\mathbb{K}^{n\times 1}$. Show that $(Y^{\perp})^{\perp} = Y$.

- We need to prove that: (i) $Y \subseteq (Y^{\perp})^{\perp}$ and (ii) $(Y^{\perp})^{\perp} \subseteq Y$
- (i) $Y \subseteq (Y^{\perp})^{\perp}$
  - Suppose $\mathbf{x} \in Y$. We know $Y^{\perp} = \{\mathbf{y} \in \mathbb{K}^{n\times 1} \,|\langle \mathbf{x}, \mathbf{y}\rangle = 0 \;\forall\, \mathbf{x} \in Y\}$
  - So $\langle \mathbf{x}, \mathbf{y}\rangle = 0 \;\forall\, \mathbf{y} \in Y^{\perp}$.
  - Thus, be definition of $(Y^{\perp})^{\perp}$, $\mathbf{x} \in Y$
  - Hence $u \in (U^{\perp})^{\perp}$
- (ii) $(Y^{\perp})^{\perp} \subseteq Y$
  - Let $\mathbf{x} \in (Y^{\perp})^{\perp} \subseteq \mathbb{K}^{n\times 1}$
  - By Projection Theorem, $\mathbf{x}$ can be written as $\mathbf{x} = \mathbf{y} + \tilde{\mathbf{y}}$ where $\mathbf{y} \in Y$ and $\tilde{\mathbf{y}} \in Y^{\perp}$
  - $\mathbf{x} - \mathbf{y} = \tilde{\mathbf{y}}$, so $\mathbf{x} - \mathbf{y} \in Y^{\perp}$.
  - Also, we had, $\mathbf{x} \in (Y^{\perp})^{\perp}$ and $\mathbf{y} \in Y \subseteq (U^{\perp})^{\perp}$ (from part i)
  - So $\mathbf{x} - \mathbf{y} \in (Y^{\perp})^{\perp}$
  - Or $\mathbf{x} - \mathbf{y} \in Y^{\perp} \cap (Y^{\perp})^{\perp}$
  - Hence $\mathbf{x} - \mathbf{y} = 0$, so $\mathbf{x} = \mathbf{y} \in Y$

Question 9

Let $\mathbf{A}$ be a self-adjoint matrix. If $\langle \mathbf{Ax}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then show that $\mathbf{A} = \mathbf{O}$.

- By Spectral Theorem of Self-Adjoint Matrices, $\mathbf{A}$ is self-adjoint $\Leftrightarrow$ $\mathbf{A}$ is unitarily diagonalizable and all eigen values are real
- That is $\mathbf{A} = \mathbf{UDU}^*$, where $\mathbf{U} = \left[\mathbf{u}_1, \ldots \mathbf{u}_n\right]$ and $\mathbf{u}_i$ are orthonormal
- Since $\langle \mathbf{Ax}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$,

$$\langle \mathbf{Au}_i, \mathbf{u}_i \rangle = \langle \lambda_i \mathbf{u}_i, \mathbf{u}_i \rangle = \overline{\lambda_i} \mathbf{u}_i{}^* \mathbf{u}_i = \lambda_i = 0$$

- Thus $\mathbf{D} = \mathbf{O} \implies \mathbf{A} = \mathbf{O}$

Deduce that if $\|\mathbf{A}^*\mathbf{x}\| = \|\mathbf{Ax}\|$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then $\mathbf{A}$ is a normal matrix

- $\|\mathbf{A}^*\mathbf{x}\| = \|\mathbf{Ax}\| \Rightarrow \langle \mathbf{A}^*\mathbf{x}, \mathbf{A}^*\mathbf{x} \rangle = \langle \mathbf{Ax}, \mathbf{Ax} \rangle$
- $\mathbf{x}^*\mathbf{AA}^*\mathbf{x} = \mathbf{x}^*\mathbf{A}^*\mathbf{Ax} \Rightarrow \mathbf{x}^*(\mathbf{AA}^*\mathbf{x} - \mathbf{A}^*\mathbf{Ax}) = 0$
- $\langle \mathbf{x}, (\mathbf{AA}^* - \mathbf{A}^*\mathbf{A})\mathbf{x} \rangle = 0$
- Check that $\mathbf{AA}^* - \mathbf{A}^*\mathbf{A}$ is self-adjoint
- Using the previous part, we get $\mathbf{AA}^* - \mathbf{A}^*\mathbf{A} = 0$, or $\mathbf{A}$ is normal

Deduce that if $\|\mathbf{Ax}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{K}^{n\times 1}$, then $\mathbf{A}$ is a unitary matrix.

- $\|\mathbf{A}^*\mathbf{x}\| = \|\mathbf{x}\| \Rightarrow \langle \mathbf{A}^*\mathbf{x}, \mathbf{A}^*\mathbf{x}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle$
- $\mathbf{x}^*\mathbf{AA}^*\mathbf{x} = \mathbf{x}^*\mathbf{x} \Rightarrow \mathbf{x}^*(\mathbf{AA}^*\mathbf{x} - \mathbf{x}) = 0$
- $\langle \mathbf{x}, (\mathbf{AA}^* - \mathbf{I})\mathbf{x}\rangle = 0$
- $\langle (\mathbf{AA}^* - \mathbf{I})\mathbf{x}, \mathbf{x}\rangle = \overline{0} = 0$
- Check that $\mathbf{AA}^* - \mathbf{I}$ is self-adjoint
- Using the first part, we get $\mathbf{AA}^* - \mathbf{I} = 0$, or $\mathbf{A}$ is normal

Question 10 (i)

Let E be a nonempty subset of $\mathbb{K}^{n\times 1}$. If E is not closed, then show that there is $\mathbf{x} \in \mathbb{K}^{n\times 1}$ such that no best approximation to $\mathbf{x}$ exists from E.

- Recall a set is closed if it contains all its limit points *i.e.* for all sequences $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in E$, $\lim_{n\to\infty} a_n = a$ and $a \in E$
- Now if E is not closed, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in E$ but $\lim_{n\to\infty} a_n = a \notin E$
- Choose $\mathbf{x}$ to be a and for contradiction assume that it has a best approximation y_0
- Since $a \notin E$, $\|a - y_0\| > 0$
- Definition of limits, $\forall\, \epsilon > 0,\ \exists N \in \mathbb{N}$ s.t. $\forall\, n \geq N,\ \|a_n - a\| < \epsilon$
- Choose $\epsilon = \|a - y_0\|$, thus we have that $\exists N \in \mathbb{N}$ s.t. $\|a_N - a\| < \epsilon = \|a - y_0\|$
- This contradicts the assumption that y_0 is the best approximation
- Thus, no best approximation of a exists

Question 10 (ii)

Let E be a nonempty subset of $\mathbb{K}^{n\times 1}$. If E is convex, then show that for every $\mathbf{x} \in \mathbb{K}^{n\times 1}$, there is at most one best approximation to $\mathbf{x}$ from E.

- For contradiction, suppose some point $\mathbf{x}$ has more than one best approximation, say y_1 and y_2 ($y_1 \neq y_2$). Thus $\|x - y_1\| = \|x - y_2\| (= d)$
- Since E is convex, $\frac{y_1+y_2}{2} \in E$
- Now $\|x - \frac{y_1+y_2}{2}\| < \frac{1}{2}\|x - y_1\| + \frac{1}{2}\|x - y_2\| < d$
- This contradicts the assumption at y_1 and y_2 are the best approximations
- Thus there must be at most one best approximation to $\mathbf{x}$

Question 11

Find $\mathbf{x} := [x_1, x_2]^\mathsf{T} \in \mathbb{R}^{2\times 1}$ such that the straight line $t = x_1 + x_2 s$ fits the data points $(-1, 2)$, $(0, 0)$, $(1, -3)$ and $(2, -5)$ best in the 'least squares' sense.

- $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}$
- $\mathcal{C}(\mathbf{A})$ is the span of $\mathbf{c}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top$ and $\mathbf{c}_2 = \begin{bmatrix} -1 & 0 & 1 & 2 \end{bmatrix}^\top$
- GSOP gives $\mathbf{u}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top$, $\mathbf{u}_2 = \frac{1}{2\sqrt{5}}\begin{bmatrix} -3 & 1 & 1 & 3 \end{bmatrix}^\top$
- Best approximation to $\mathbf{b}$ from $\mathcal{C}(\mathbf{A})$ is $\mathbf{a} = \langle \mathbf{u}_1, \mathbf{b}\rangle \mathbf{u}_1 + \langle \mathbf{u}_2, \mathbf{b}\rangle \mathbf{u}_2$
- $\mathbf{a} = -3\mathbf{u}_1 - \frac{12}{\sqrt{5}}\mathbf{u}_2 = \begin{bmatrix} 2.1 & -0.3 & -2.7 & -5.1 \end{bmatrix}^\top$
- Solve $\mathbf{Ax} = \mathbf{a}$
- $x_1 - x_2 = 2.1$, $x_1 = -0.3$, $x_1 + x_2 = -2.7$, $x_1 + 2x_2 = -5.1$
- $x_1 = -0.3$, $x_2 = -2.4$ or $\mathbf{x} = \begin{bmatrix} -0.3 \\ -2.4 \end{bmatrix}$

Question 12

Let $Q(x_1, \ldots, x_n) := \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} x_k \overline{x}_j$, where $\alpha_{jk} \in \mathbb{C}$, be a **complex quadratic form**. If the quadratic form $Q(x_1, \ldots, x_n)$ takes values in $\mathbb{R}$ for all $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}} \in \mathbb{C}^{n\times 1}$, then show that there is a unique self-adjoint matrix $\mathbf{A}$ such that

$$Q(x_1, \ldots, x_n) = \mathbf{x}^* \mathbf{A}\, \mathbf{x} \quad \text{for all } \mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}} \in \mathbb{C}^{n\times 1}$$

- $Q(x_1, \ldots, x_n) = \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} x_k \overline{x}_j = \sum_{j=1}^{n} (\sum_{k=1}^{n} \alpha_{jk} x_k) \overline{x}_j$
- $= \mathbf{x}^* \begin{bmatrix} \sum_{k=1}^{n} \alpha_{1k} x_k \\ \vdots \\ \sum_{k=1}^{n} \alpha_{jk} x_k \\ \vdots \\ \sum_{k=1}^{n} \alpha_{nk} x_k \end{bmatrix} = \mathbf{x}^* \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \mathbf{x}$

If the quadratic form $Q(x_1, \ldots, x_n)$ takes values in $\mathbb{R}$ for all $\mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}} \in \mathbb{C}^{n\times 1}$, then show that there is a unique self-adjoint matrix $\mathbf{A}$ such that

$$Q(x_1, \ldots, x_n) = \mathbf{x}^* \mathbf{A}\, \mathbf{x} \quad \text{for all } \mathbf{x} := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^{\mathsf{T}} \in \mathbb{C}^{n\times 1}$$

- $\mathbf{A} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}$ is one possible $\mathbf{A}$
- Now, Q only takes real values, so $Q(x_1, \ldots, x_n) = \overline{Q(x_1, \ldots, x_n)}$
- $\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} x_k \overline{x}_j = \sum_{j=1}^{n} \sum_{k=1}^{n} \overline{\alpha_{jk}}\, \overline{x_k} x_j$
- Changing index variables on the RHS,
- $\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk} x_k \overline{x}_j = \sum_{k=1}^{n} \sum_{j=1}^{n} \overline{\alpha_{kj}}\, \overline{x_j} x_k$
- Thus, we can have $\alpha_{jk} = \overline{\alpha_{kj}}$ which combined with the $\mathbf{A}$ above gives us a self-adjoint matrix

- It remains to be shown that $\mathbf{A}$ is unique
- For contradiction, let $\mathbf{A}$ and $\mathbf{B}$ be two self-adjoint matrices representing Q
- $\mathbf{x}^*\mathbf{A}\mathbf{x} = \mathbf{x}^*\mathbf{B}\mathbf{x} \implies \langle(\mathbf{A} - \mathbf{B})\mathbf{x}, \mathbf{x}\rangle = 0 \; \forall\, \mathbf{x} \in \mathbb{C}^{n\times 1}$
- Noting that $\mathbf{A} - \mathbf{B}$ is self adjoint and using 7.9 gives us $\mathbf{A} = \mathbf{B}$