MA 106 : Linear Algebra

Tutorial 5

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Question 1(i)

Find all eigenvalues, and their geometric as well as algebraic multiplicities of the following matrices. Are they diagonalizable? If so, find invertible $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

$\mathbf{A} := \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$

- $p_{\mathbf{A}}(t) = \det \begin{bmatrix} 5-t & -1 \\ 1 & 3-t \end{bmatrix}$
- $= (5-t)(3-t) + 1 = t^2 - 8t + 16$
- $= (t-4)^2$
- $\mathbf{A} - 4\mathbf{I} = \det \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \to \text{rank}(\mathbf{A} - 4\mathbf{I}) = 1$
- Eigenvalue: 4 with algebraic multiplicity 2 and geometric multiplicity $= \text{nullity}(\mathbf{A} - 4\mathbf{I}) = 2 - 1 = 1$
- Not diagonalizable

Question 1(ii)

$$\mathbf{A} := \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

- $p_{\mathbf{A}}(t) = \det \begin{bmatrix} 3-t & 2 & 1 & 0 \\ 0 & 1-t & 0 & 1 \\ 0 & 2 & -1-t & 0 \\ 0 & 0 & 0 & 1/2-t \end{bmatrix}$
- $= (3-t) \begin{bmatrix} 1-t & 0 & 1 \\ 2 & -1-t & 0 \\ 0 & 0 & 1/2-t \end{bmatrix}$
- $= (3-t)(1/2-t) \begin{bmatrix} 1-t & 0 \\ 2 & -1-t \end{bmatrix} = (3-t)(1/2-t)(1-t)(-1-t)$
- Eigenvalues: $\{3, 1/2, 1, -1\}$ all with algebraic and geometric multiplicity 1
- Diagonalizable

- $\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$ has eigenvalues: {3, 1/2, 1, -1}
- For $\lambda = 3$,
  - $\mathbf{A} - 3\mathbf{I} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & -5/2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5/2 \end{bmatrix} \rightarrow$

    $\begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Basic solution $= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top$
- Similarly, check that for $\lambda = 1/2$, $\begin{bmatrix} 8/3 & -2 & -8/3 & 1 \end{bmatrix}^\top$; $\lambda = 1$, $\begin{bmatrix} -3/2 & 1 & 1 & 0 \end{bmatrix}^\top$; $\lambda = -1$, $\begin{bmatrix} -1/4 & 0 & 1 & 0 \end{bmatrix}^\top$

- $\mathbf{P} = \begin{bmatrix} 1 & 8/3 & -3/2 & -1/4 \\ 0 & -2 & 1 & 0 \\ 0 & -8/3 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Question 1(iii)

$$\mathbf{A} := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

- $p_{\mathbf{A}}(t) = \det \begin{bmatrix} 2-t & 1 & 0 \\ 0 & 2-t & 1 \\ 0 & 0 & 2-t \end{bmatrix} = (2-t)^3$
- $\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(\mathbf{A} - 2\mathbf{I}) = 2$
- Eigenvalue: 4 with algebraic multiplicity 2 and geometric multiplicity $= \text{nullity}(\mathbf{A} - 2\mathbf{I}) = 3 - 2 = 1$
- Not diagonalizable

Question 2

Let $\mathbf{A} := \begin{bmatrix} 2 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}$. Find a necessary and sufficient condition on a, b, c for $\mathbf{A}$ to be diagonalizable.

- $p_{\mathbf{A}}(t) = \det \begin{bmatrix} 2-t & a & b \\ 0 & 1-t & c \\ 0 & 0 & 2-t \end{bmatrix} = (2-t)^2(1-t)$
- For $\mathbf{A}$ to be diagonalizable, we need the algebraic and geometric multiplicities of the eigenvalues to match
- For $\lambda = 1$, this is always true (why?).
- For $\lambda = 2$, we need its geometric multiplicity to be 2, or nullity$(\mathbf{A} - 2\mathbf{I}) = 2$, or rank$(\mathbf{A} - 2\mathbf{I}) = 1$
- $\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -c \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -c \\ 0 & 0 & b+ac \\ 0 & 0 & 0 \end{bmatrix}$
- So, we need $b + ac = 0$

- $\mathbf{A} - \mathbf{I} = \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & b \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
- $\text{rank}(\mathbf{A} - \mathbf{I}) = 2 \Rightarrow \text{nullity}(\mathbf{A} - \mathbf{I}) = 1 \Rightarrow \lambda = 1$ has geometric multiplicity 1

Question 4

Let $\lambda \in \mathbb{K}$. Show that λ is an eigenvalue of $\mathbf{A}$ if and only if $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}^*$, but their eigenvectors can be very different

- $\det \mathbf{A}^* = \det \overline{\mathbf{A}}^\top = \det \overline{\mathbf{A}} = \overline{\det \mathbf{A}}$ (Prove by induction)
- Outline of proof:
  - Base case: $\det \begin{bmatrix} x \end{bmatrix}^* = \det \begin{bmatrix} \overline{x} \end{bmatrix} = \overline{x} = \overline{\det \begin{bmatrix} x \end{bmatrix}}$
  - Let $\mathbf{B} = \overline{\mathbf{A}}$ and let N_{ij} be the minors of $\mathbf{A}$
  - $\det \mathbf{B} = \sum_n (-1)^{1+i} b_{1i} N_{1i}$
  - $= \sum_n (-1)^{1+i} \overline{a_{1i}}\, \overline{M_{1i}}$ (minors are determinants of a smaller size)
  - $= \sum_n (-1)^{1+i} \overline{a_{1i} M_{1i}} = \overline{\sum_n (-1)^{1+i} a_{1i} M_{1i}} = \overline{\det \mathbf{A}}$
- Thus we have $\det \mathbf{A}^* = \overline{\det \mathbf{A}}$
- We have $\det(\mathbf{A} - \lambda \mathbf{I}) = \overline{\overline{\det(\mathbf{A} - \lambda \mathbf{I})}} = \overline{\det((\mathbf{A} - \lambda \mathbf{I})^*)}$ $= \overline{\det(\mathbf{A}^* - \overline{\lambda} \mathbf{I})}$
- Thus $\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \iff \det(\mathbf{A}^* - \overline{\lambda} \mathbf{I}) = 0$

Question 5

Let $\mathbf{A} \in \mathbb{K}^{n \times n}$. Show that 0 is an eigenvalue of $\mathbf{A}$ if and only if 0 is an eigenvalue of $\mathbf{A}^*\mathbf{A}$, and its geometric multiplicity is the same.

- $(\Longrightarrow)$
  - 0 is an eigenvalue of $\mathbf{A}$
  - $\mathbf{Ax} = 0 \Rightarrow \mathbf{A}^*\mathbf{Ax} = 0$
  - 0 is an eigenvalue of $\mathbf{A}^*\mathbf{A}$
- $(\Longleftarrow)$
  - 0 is an eigenvalue of $\mathbf{A}^*\mathbf{A}$
  - $\mathbf{A}^*\mathbf{Ax} = 0 \Rightarrow \mathbf{x}^*\mathbf{A}^*\mathbf{Ax} = 0$
  - $(\mathbf{Ax})^*\mathbf{Ax} = 0 \Rightarrow \mathbf{Ax} = 0$ Why?
  - 0 is an eigenvalue of $\mathbf{A}$
- Thus we have shown that 0 is an eigenvalue of $\mathbf{A}$ if and only if 0 is an eigenvalue of $\mathbf{A}^*\mathbf{A}$

Let $\mathbf{A} \in \mathbb{K}^{n\times n}$. Show that 0 is an eigenvalue of $\mathbf{A}$ if and only if 0 is an eigenvalue of $\mathbf{A}^*\mathbf{A}$, and its geometric multiplicity is the same.

- Geometric multiplicity $= \dim \mathcal{N}(\mathbf{A} - \lambda\mathbf{I}) = \dim \mathcal{N}(\mathbf{A})$ (for $\lambda = 0$)
- Thus to prove equal geometric multiplicity of 0, we need to show that $\dim \mathcal{N}(\mathbf{A}) = \dim \mathcal{N}(\mathbf{A}^*\mathbf{A})$
- We instead prove a stronger statement: $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^*\mathbf{A})$
  - $\mathbf{x} \in \mathcal{N}(\mathbf{A}) \Leftrightarrow \mathbf{Ax} = 0 \iff \mathbf{A}^*\mathbf{Ax} = 0 \Leftrightarrow \mathbf{x} \in \mathcal{N}(\mathbf{A}^*\mathbf{A})$
  - Above statement follows from the previous part
  - $\mathbf{x} \in \mathcal{N}(\mathbf{A}) \Leftrightarrow \mathbf{x} \in \mathcal{N}(\mathbf{A}^*\mathbf{A}) \Longrightarrow \mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^*\mathbf{A})$

Deduce $\operatorname{rank} \mathbf{A}^*\mathbf{A} = \operatorname{rank} \mathbf{A}$

- We have shown that $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^*\mathbf{A})$
- $\dim \mathcal{N}(\mathbf{A}) = \dim \mathcal{N}(\mathbf{A}^*\mathbf{A}) \Rightarrow \text{nullity}(\mathbf{A}) = \text{nullity}(\mathbf{A}^*\mathbf{A})$
- By rank-nullity theorem, $\operatorname{rank} \mathbf{A}^*\mathbf{A} = \operatorname{rank} \mathbf{A}$ follows (both matrices have the same number of columns)

Question 7

A square matrix $\mathbf{A} := [a_{jk}]$ is called **strictly diagonally dominant** if $|a_{jj}| > \sum_{k \neq j} |a_{jk}|$ for each $j = 1, \ldots, n$. If $\mathbf{A}$ strictly diagonally dominant, show that $\mathbf{A}$ is invertible.

- Showing $\mathbf{A}$ is invertible is equivalent to showing nullity $\mathbf{A} = 0$
- Or $\mathbf{Ax} = 0$ does not have any non-zero solution or 0 is not an eigenvalue of $\mathbf{A}$

Recall: Gerschgorin Theorem

Every eigenvalue of $\mathbf{A} \in \mathbb{K}^{n \times n}$ belongs in one of its Gerschgorin disks.
If $r_j := \sum_{k \neq j} |a_{jk}|$, the Gerschgorin disks are given by
$D(a_{jj}, r_j) := \{a \in \mathbb{K} : |a - a_{jj}| \leq r_j\}$

- Since for strictly diagonally dominant matrix $\mathbf{A}$, $|a_{jj}| > r_j$, by Gerschgorin Theorem, none of the disks can contain the origin (distance of center from origin $>$ radius)
- Thus 0 cannot be a eigenvalue of $\mathbf{A}$

Question 8

Let $\mathbf{A} \in \mathbb{K}^{n\times n}$. Define $\alpha_2 := \max\{\|\mathbf{Ax}\| : \|\mathbf{x}\| = 1\}$,
$\alpha_\infty := \max\left\{\sum_{k=1}^{n} |a_{jk}| : j = 1, \ldots, n\right\}$ and
$\alpha_1 := \max\left\{\sum_{j=1}^{n} |a_{jk}| : k = 1, \ldots, n\right\}$, where $\mathbf{A} := [a_{jk}]$.
Show that $|\lambda| \leq \min\{\alpha_2, \alpha_\infty, \alpha_1\}$ for every eigenvalue λ.

- $|\lambda| \leq \min\{\alpha_2, \alpha_\infty, \alpha_1\} \iff |\lambda| \leq \alpha_2$ and $|\lambda| \leq \alpha_\infty$ and $|\lambda| \leq \alpha_2$
- First we show that $|\lambda| \leq \alpha_2$
  - Let $\lambda_{\max}$ be the eigenvalue with maximum absolute value *i.e.* $|\lambda_{\max}| \geq |\lambda_i| \forall i \in \{1, 2 \ldots, n\}$
  - Thus there exists a vector $\mathbf{x}$ with $\|\mathbf{x}\| = 1$ such that $\mathbf{Ax} = \lambda_{\max}\mathbf{x}$
  - $\|\mathbf{Ax}\| = \|\lambda_{\max}\mathbf{x}\| = |\lambda_{\max}|\|\mathbf{x}\| = |\lambda_{\max}|$
  - Which gives us $\alpha_2 = \max\{\|\mathbf{Ax}\| : \|\mathbf{x}\| = 1\} \geq |\lambda_{\max}|$ (Why?)
  - $|\lambda_{\max}| \leq \alpha_2 \Rightarrow |\lambda_i| \leq \alpha_2$ for all i (Why?)
- Next we show that $|\lambda| \leq \alpha_\infty$
  - Let j' be the value of j at which $\sum_k |a_{jk}|$ attains it maximum
  - By Gershgorin Theorem, $|\lambda - a_{j'j'}| \leq \sum_{k,\, k\neq j'} |a_{j'k}|$
  - $|\lambda| - |a_{j'j'}| \leq |\lambda - a_{j'j'}| \leq \sum_{k,\, k\neq j'} |a_{j'k}|$ (Triangle Inequality)
  - $|\lambda| \leq \sum_{k,\, k\neq j'} |a_{j'k}| + |a_{j'j'}| = \alpha_\infty$

Let $\mathbf{A} \in \mathbb{K}^{n \times n}$. Define $\alpha_2 := \max\{\|\mathbf{Ax}\| : \|\mathbf{x}\| = 1\}$,
$\alpha_\infty := \max\left\{ \sum_{k=1}^{n} |a_{jk}| : j = 1, \ldots, n\right\}$ and
$\alpha_1 := \max\left\{ \sum_{j=1}^{n} |a_{jk}| : k = 1, \ldots, n\right\}$, where $\mathbf{A} := [a_{jk}]$.
Show that $|\lambda| \leq \min\{\alpha_2, \alpha_\infty, \alpha_1\}$ for every eigenvalue λ.

- Finally, we show that $|\lambda| \leq \alpha_1$
  - Observe that $|\lambda| \leq \alpha_\infty$ is the maximum row sum and $|\lambda| \leq \alpha_1$ is the maximum column sum
  - If we show that $\mathbf{A}$ and $\mathbf{A}^\top$ have the same eigenvalues, we can apply Gershgorin Theorem as in the previous part and be done
  - $\det(\mathbf{A} - \lambda \mathbf{I}) = \det((\mathbf{A} - \lambda \mathbf{I})^\top) = \det(\mathbf{A}^\top - \lambda \mathbf{I})$
  - $\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \Leftrightarrow \det(\mathbf{A}^\top - \lambda \mathbf{I}) = 0$ or both $\mathbf{A}$ and $\mathbf{A}^\top$ have the same eigenvalues
  - Can you complete the proof?

Question 9

Let $\mathbf{x}, \mathbf{y} \in \mathbb{K}^{n\times 1}$. Prove $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$.

- $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})^*(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})^*(\mathbf{x}-\mathbf{y})$
- $= (\mathbf{x}^*+\mathbf{y}^*)(\mathbf{x}+\mathbf{y}) + (\mathbf{x}^*-\mathbf{y}^*)(\mathbf{x}-\mathbf{y})$
- $= \mathbf{x}^*\mathbf{x} + \mathbf{y}^*\mathbf{y} + \mathbf{x}^*\mathbf{y} + \mathbf{y}^*\mathbf{x} + \mathbf{x}^*\mathbf{x} + \mathbf{y}^*\mathbf{y} - \mathbf{x}^*\mathbf{y} - \mathbf{y}^*\mathbf{x}$
- $= 2\mathbf{x}^*\mathbf{x} + 2\mathbf{y}^*\mathbf{y} = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$

If $\mathbf{x}$ and $\mathbf{y}$ are nonzero, prove $\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$, where $\theta \in [0,\pi]$ is defined to be $\cos^{-1}(\Re\langle\mathbf{x},\mathbf{y}\rangle/\|\mathbf{x}\|\|\mathbf{y}\|)$.

- $\|\mathbf{x}-\mathbf{y}\|^2 = (\mathbf{x}-\mathbf{y})^*(\mathbf{x}-\mathbf{y}) = \mathbf{x}^*\mathbf{x} + \mathbf{y}^*\mathbf{y} - \mathbf{x}^*\mathbf{y} - \mathbf{y}^*\mathbf{x}$
- $= \mathbf{x}^*\mathbf{x} + \mathbf{y}^*\mathbf{y} - \langle\mathbf{x},\mathbf{y}\rangle - \langle\mathbf{y},\mathbf{x}\rangle = \mathbf{x}^*\mathbf{x} + \mathbf{y}^*\mathbf{y} - \langle\mathbf{x},\mathbf{y}\rangle - \overline{\langle\mathbf{x},\mathbf{y}\rangle}$
- $= \mathbf{x}^*\mathbf{x} + \mathbf{y}^*\mathbf{y} - 2\Re(\mathbf{x}^*\mathbf{y}) = \mathbf{x}^*\mathbf{x} + \mathbf{y}^*\mathbf{y} - 2\Re\langle\mathbf{x},\mathbf{y}\rangle$
  $= \mathbf{x}^*\mathbf{x} + \mathbf{y}^*\mathbf{y} - 2\|\mathbf{x}\|\|\mathbf{y}\|\Re\langle\mathbf{x},\mathbf{y}\rangle/\|\mathbf{x}\|\|\mathbf{y}\|$
- Now, $\frac{|\Re\langle\mathbf{x},\mathbf{y}\rangle|}{\|\mathbf{x}\|\|\mathbf{y}\|} \le \frac{|\langle\mathbf{x},\mathbf{y}\rangle|}{\|\mathbf{x}\|\|\mathbf{y}\|} \le 1$ (Schwarz inequality)
- $\Rightarrow -1 \le \frac{\Re\langle\mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|\|\mathbf{y}\|} \le 1$. So, we can conclude that $\theta \in [0,\pi]$ exist

$\overline{\langle\mathbf{x},\mathbf{y}\rangle} = \overline{\mathbf{x}^*\mathbf{y}} = \overline{\mathbf{x}^*}\,\overline{\mathbf{y}} = \overline{\overline{\mathbf{x}}}^\top\overline{\mathbf{y}} = \mathbf{x}^\top\overline{\mathbf{y}} = \overline{\mathbf{y}}^\top\mathbf{x} = \mathbf{y}^*\mathbf{x} = \langle\mathbf{y},\mathbf{x}\rangle$

Question 3 and 6

- See the OneNote notebook in MS Teams

Tutorial 6 Question 1

Orthonormalize the following ordered subsets of $\mathbb{K}^{4\times 1}$

(i) $(\mathbf{e}_1, \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)$

- $\mathbf{y}_1 = \mathbf{e}_1$
- $\mathbf{y}_2 = \mathbf{e}_1+\mathbf{e}_2 - \frac{\langle \mathbf{e}_1, \mathbf{e}_1+\mathbf{e}_2\rangle}{\langle \mathbf{e}_1, \mathbf{e}_1\rangle}\mathbf{e}_1 = \mathbf{e}_2$
- $\mathbf{y}_3 = \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3 - \frac{\langle \mathbf{e}_1, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3\rangle}{\langle \mathbf{e}_1, \mathbf{e}_1\rangle}\mathbf{e}_1 - \frac{\langle \mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3\rangle}{\langle \mathbf{e}_2, \mathbf{e}_2\rangle}\mathbf{e}_2 = \mathbf{e}_3$
- $\mathbf{y}_4 = \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4 - \frac{\langle \mathbf{e}_1, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4\rangle}{\langle \mathbf{e}_1, \mathbf{e}_1\rangle}\mathbf{e}_1 - \frac{\langle \mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4\rangle}{\langle \mathbf{e}_2, \mathbf{e}_2\rangle}\mathbf{e}_2 - \frac{\langle \mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4\rangle}{\langle \mathbf{e}_3, \mathbf{e}_3\rangle}\mathbf{e}_3 = \mathbf{e}_4$

(ii) $(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4, -\mathbf{e}_1+\mathbf{e}_2, -\mathbf{e}_1+\mathbf{e}_3, -\mathbf{e}_1+\mathbf{e}_4)$

- $\mathbf{y}_1 = \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4$
- $\mathbf{y}_2 = -\mathbf{e}_1+\mathbf{e}_2 - \frac{\langle \mathbf{y}_1, -\mathbf{e}_1+\mathbf{e}_2\rangle}{\langle \mathbf{y}_1, \mathbf{y}_1\rangle}\mathbf{y}_1 = -\mathbf{e}_1+\mathbf{e}_2$
- $\mathbf{y}_3 = -\mathbf{e}_1+\mathbf{e}_3 - 0\cdot\mathbf{y}_1 - \frac{\langle -\mathbf{e}_1+\mathbf{e}_3, -\mathbf{e}_1+\mathbf{e}_2\rangle}{\langle -\mathbf{e}_1+\mathbf{e}_2, -\mathbf{e}_1+\mathbf{e}_2\rangle}(-\mathbf{e}_1+\mathbf{e}_2) = -\mathbf{e}_1+\mathbf{e}_3 - \frac{-\mathbf{e}_1+\mathbf{e}_2}{2} = -\frac{1}{2}\mathbf{e}_1 - \frac{1}{2}\mathbf{e}_2 + \mathbf{e}_3$
- $\mathbf{y}_4 = -\mathbf{e}_1+\mathbf{e}_4 - 0\cdot\mathbf{y}_1 - \frac{\langle -\mathbf{e}_1+\mathbf{e}_4, \mathbf{y}_2\rangle}{\langle \mathbf{y}_2, \mathbf{y}_2\rangle}\mathbf{y}_2 - \frac{\langle -\mathbf{e}_1+\mathbf{e}_4, \mathbf{y}_3\rangle}{\langle \mathbf{y}_3, \mathbf{y}_3\rangle}\mathbf{y}_3 = -\mathbf{e}_1+\mathbf{e}_4 - \frac{1}{2}\mathbf{y}_2 - \frac{1/2}{3/2}\mathbf{y}_3 = -\frac{1}{3}\mathbf{e}_1 - \frac{1}{3}\mathbf{e}_2 - \frac{1}{3}\mathbf{e}_3 + \mathbf{e}_4$
- $\left(\frac{\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4}{2}, \frac{-\mathbf{e}_1+\mathbf{e}_2}{\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}\mathbf{e}_1 - \frac{\sqrt{3}}{2\sqrt{2}}\mathbf{e}_2 + \frac{\sqrt{3}}{\sqrt{2}}\mathbf{e}_3, -\frac{\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3}{2\sqrt{3}} + \frac{\sqrt{3}}{2}\mathbf{e}_4\right)$

Tutorial 6 Question 2

Use the Gram-Schmidt Orthogonalization Process to orthonormalize the ordered subset

$$\left(\begin{bmatrix}1 & -1 & 2 & 0\end{bmatrix}^{\mathsf{T}}, \begin{bmatrix}1 & 1 & 2 & 0\end{bmatrix}^{\mathsf{T}}, \begin{bmatrix}3 & 0 & 0 & 1\end{bmatrix}^{\mathsf{T}}\right)$$

and obtain an ordered orthonormal set $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

- $\mathbf{y}_1 = \begin{bmatrix}1 & -1 & 2 & 0\end{bmatrix}^{\mathsf{T}}$
- $\mathbf{y}_2 = \begin{bmatrix}1 & 1 & 2 & 0\end{bmatrix}^{\mathsf{T}} - \frac{4}{6}\begin{bmatrix}1 & -1 & 2 & 0\end{bmatrix}^{\mathsf{T}} = \begin{bmatrix}\frac{1}{3} & \frac{5}{3} & \frac{2}{3} & 0\end{bmatrix}^{\mathsf{T}}$
- $\mathbf{y}_3 = \begin{bmatrix}3 & 0 & 0 & 1\end{bmatrix}^{\mathsf{T}} - \frac{3}{6}\begin{bmatrix}1 & -1 & 2 & 0\end{bmatrix}^{\mathsf{T}} - \frac{1}{10/3}\begin{bmatrix}\frac{1}{3} & \frac{5}{3} & \frac{2}{3} & 0\end{bmatrix}^{\mathsf{T}} = \begin{bmatrix}\frac{12}{5} & 0 & -\frac{6}{5} & 1\end{bmatrix}^{\mathsf{T}}$

- $\mathbf{u}_1 = \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|} = \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & 0 \end{bmatrix}^T$
- $\mathbf{u}_2 = \frac{\mathbf{y}_2}{\|\mathbf{y}_2\|} = \begin{bmatrix} \frac{\sqrt{30}}{30} & \frac{\sqrt{30}}{6} & \frac{\sqrt{30}}{15} & 0 \end{bmatrix}^T$
- $\mathbf{u}_3 = \frac{\mathbf{y}_3}{\|\mathbf{y}_3\|} = \begin{bmatrix} \frac{12\sqrt{205}}{205} & 0 & -\frac{6\sqrt{205}}{205} & \frac{\sqrt{205}}{41} \end{bmatrix}^T$

Find $\mathbf{u}_4$ such that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ is an ordered orthonormal basis for $\mathbb{K}^{4\times 1}$

- We want to $\mathbf{u}_4$ such that $\langle \mathbf{u}_1, \, \mathbf{u}_4 \rangle = \langle \mathbf{u}_3, \, \mathbf{u}_4 \rangle = \langle \mathbf{u}_3, \, \mathbf{u}_4 \rangle = 0$
- Find the basic solution of (or directly use $\mathbf{y}_i$ instead of $\mathbf{u}_1$),

$$\begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & 0 \\ \frac{\sqrt{30}}{30} & \frac{\sqrt{30}}{6} & \frac{\sqrt{30}}{15} & 0 \\ \frac{12\sqrt{205}}{205} & 0 & -\frac{6\sqrt{205}}{205} & \frac{\sqrt{205}}{41} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- It turns out to be $\mathbf{y}_4 = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{6} & 1 \end{bmatrix}^\top$
- $\mathbf{u}_4 = \begin{bmatrix} -\frac{2}{\sqrt{41}} & 0 & \frac{1}{\sqrt{41}} & \frac{6}{\sqrt{41}} \end{bmatrix}^\top$

Express the vector $\mathbf{x} = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$ as a linear combination of these four basis vectors.

- $\mathbf{u}_1 = \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & 0 \end{bmatrix}^T$
- $\mathbf{u}_2 = \begin{bmatrix} \frac{\sqrt{30}}{30} & \frac{\sqrt{30}}{6} & \frac{\sqrt{30}}{15} & 0 \end{bmatrix}^T$
- $\mathbf{u}_3 = \begin{bmatrix} \frac{12\sqrt{205}}{205} & 0 & -\frac{6\sqrt{205}}{205} & \frac{\sqrt{205}}{41} \end{bmatrix}^T$
- $\mathbf{u}_4 = \begin{bmatrix} -\frac{2\sqrt{41}}{41} & 0 & \frac{\sqrt{41}}{41} & \frac{6\sqrt{41}}{41} \end{bmatrix}^T$
- $\mathbf{x} = P_{\mathbf{u}_1}(\mathbf{x}) + P_{\mathbf{u}_2}(\mathbf{x}) + P_{\mathbf{u}_3}(\mathbf{x}) + P_{\mathbf{u}_4}(\mathbf{x})$
- $\mathbf{x} = \langle \mathbf{u}_1, \mathbf{x} \rangle \mathbf{u}_1 + \langle \mathbf{u}_2, \mathbf{x} \rangle \mathbf{u}_2 + \langle \mathbf{u}_3, \mathbf{x} \rangle \mathbf{u}_3 + \langle \mathbf{u}_4, \mathbf{x} \rangle \mathbf{u}_4 \; (\langle \mathbf{u}, \mathbf{u} \rangle = 1)$
- $\mathbf{x} = \frac{2\sqrt{6}}{3}\mathbf{u}_1 - \frac{1\sqrt{30}}{15}\mathbf{u}_2 + \frac{\sqrt{205}}{205}\mathbf{u}_3 - \frac{7\sqrt{41}}{41}\mathbf{u}_4$