MA 106 : Linear Algebra

Tutorial 3

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Question 6

Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$

- $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ (Since $\det \mathbf{A} = \det \mathbf{A}^\top$)
- $= \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$ (Addition of multiple of a row to another)
- $= (b-a)(c-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{bmatrix}$ (Why?)
- $= (b-a)(c-a) \det \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{bmatrix}$ ($\det \mathbf{A} = \det \mathbf{A}^\top$)

- $= (b-a)(c-a)\det \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{bmatrix}$
- $R_3 - aR_2,\ R_2 - aR_1$
- $= (b-a)(c-a)\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & b & c \end{bmatrix}$ $(\det \mathbf{A} = \det \mathbf{A}^\top)$
- Expand along first row,
- $= (b-a)(c-a)\det \begin{bmatrix} 1 & 1 \\ b & c \end{bmatrix}$
- $= (b-a)(c-a)(c-b)$

Also, prove an analogous formula for a determinant of order n, known as the **Vandermonde determinant**.

- Observe that $\det \begin{bmatrix} 1 & 1 \\ b & c \end{bmatrix}$ is the Vandermonde det of order 2

- $V_n = \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & & & \\ a_1^n & a_2^n & \dots & a_n^n \end{bmatrix}$
- With the same operations as in the previous part, verify that
- $\det \begin{bmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & & & \\ a_1^n & a_2^n & \dots & a_n^n \end{bmatrix} = \prod_{i=2}^{n} (a_i - a_1) \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ a_2 & a_3 & \dots & a_n \\ \vdots & & & \\ a_2^{n-1} & a_3^{n-1} & \dots & a_n^{n-1} \end{bmatrix}$
- Thus, we have $V_n = \prod_{i=2}^{n} (a_i - a_1) V_{n-1}$. Using this and knowing $V_1 = 1$, check that
- $V_n = \prod_{1 \le i < j \le n} (a_j - a_i)$

Question 7

For $n \in \mathbb{N}$,

prove that
$$D_n = \det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ & & & & . & & \\ & & & . & & & \\ & & . & & & & \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} = (-1)^{n(n-1)/2}$$

- Expanding along the first row, we have
- $D_n = (-1)^{1+n} D_{n-1}$ $\quad$ ($(-1)^{1+n}$ since 1^{st} row and n^{th} column)
- $D_n = (-1)^{n+1}(-1)^{1+n-1} D_{n-2}$
- $D_n = (-1)^{n+1}(-1)^{n} \dots (-1)^{3} D_1$
- Since $D_1 = \det \begin{bmatrix} 1 \end{bmatrix} = 1$ and $(-1)^2 = 1$, we have
- $D_n = (-1)^{n-1}(-1)^{n-2} \dots (-1)^{1}$
- $D_n = (-1)^{n(n-1)/2}$

Question 8

For $n \in \mathbb{N}$, prove that

$$\det \begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 2 & 3 & \dots & n-1 & n \\ 3 & 3 & 3 & \dots & n-1 & n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n-1 & n-1 & n-1 & \dots & n-1 & n \\ n & n & n & \dots & n & n \end{bmatrix} = (-1)^{n+1} n.$$

- $R_1 - R_2$, $R_2 - R_3 \dots$
- $$\det \begin{bmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & -1 & 0 & \dots & 0 & 0 \\ -1 & -1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & -1 & -1 & \dots & -1 & 0 \\ n & n & n & \dots & n & n \end{bmatrix}$$
- Triangular Matrix, so $(-1)^{n-1} n = (-1)^{n+1} n$

Question 9

Find rank **A** using determinants, where **A** is

$$\text{(i)} \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix}, \text{(ii)} \begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix}.$$

Verify by transforming **A** to a REF.

- $\det \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix} = -2 \cdot 15 - 3 \cdot 10 = -60$ implies that the rank is 3
- Row reduction is

$$\begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 5 \\ 0 & 2 & -3 \\ -3 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 5 \\ 0 & 2 & -3 \\ 0 & 5 & 7.5 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & 15 \end{bmatrix}.$$

$$\text{(ii)} \begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix}$$

- $\det \begin{bmatrix} 4 & 3 \\ -8 & -6 \end{bmatrix} = 0$
- $\det \begin{bmatrix} 4 & 3 \\ 16 & 12 \end{bmatrix} = 0$
- $\det \begin{bmatrix} -8 & -6 \\ 16 & 12 \end{bmatrix} = 0$
- All 2×2 subdeterminants vanish
- there are nonzero entries, hence the rank is 1
- Row reduction is $\begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix} \to \begin{bmatrix} 4 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Tut 4 Question 1

Find the value(s) of α for which Cramer's rule is applicable. For the remaining value(s) of α, find the number of solutions, if any.

$$\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 20 \\ x & + & 3y & + & z & = & 13 \\ x & + & 6y & + & \alpha z & = & \alpha. \end{array}$$

- Cramer Rule is applicable if matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & \alpha \end{bmatrix}$ is invertible
- $\det \mathbf{A} = \det \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & \alpha \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 4 & \alpha - 3 \end{bmatrix} = \alpha + 5$
- Which is non zero for $\alpha \neq -5$

- For $\alpha = -5$,

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 1 & 3 & 1 & 13 \\ 1 & 6 & -5 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 4 & -8 & -25 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

- No solution

Tut 4 Question 2

Find the cofactor matrix $\mathbf{C}$ of the matrix $\mathbf{A}$, and verify $\mathbf{C}^\mathsf{T}\mathbf{A} = (\det \mathbf{A})\mathbf{I} = \mathbf{A}\mathbf{C}^\mathsf{T}$. If $\det \mathbf{A} \neq 0$, find $\mathbf{A}^{-1}$, where $\mathbf{A}$ is

$$\text{(i)} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{(ii)} \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \text{(iii)} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - $M_{11} = d,\ M_{12} = c,\ M_{21} = b,\ M_{22} = a$
  - $\mathbf{C} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$
  - $\mathbf{C}^\top \mathbf{A} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (\det \mathbf{A})\mathbf{I}$
  - $\mathbf{A}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

- $M_{11} = 0, M_{12} = 0, M_{13} = 4, M_{21} = -10, M_{22} = 0, M_{23} = 0, M_{31} = 0, M_{32} = -10, M_{33} = -18$
- $\mathbf{C} = \begin{bmatrix} 0 & 0 & 4 \\ 10 & 0 & 0 \\ 0 & 10 & -18 \end{bmatrix}$
- $\det \mathbf{A} = 0 \times 0 - 9 \times 0 + 5 \times 4 = 20$
- $\mathbf{C}^\top \mathbf{A} = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 10 \\ 4 & 0 & -18 \end{bmatrix} \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = (\det \mathbf{A})\mathbf{I}$
- $\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \\ 1/5 & 0 & -9/10 \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

- $M_{11} = \frac{1}{15} - \frac{1}{16} = \frac{1}{240}$, $M_{22} = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$, $\cdots$
- $\mathbf{M} = \begin{bmatrix} 1/240 & 1/60 & 1/72 \\ 1/60 & 4/45 & 1/12 \\ 1/72 & 1/12 & 1/12 \end{bmatrix}$
- $\mathbf{C} = \begin{bmatrix} 1/240 & -1/60 & 1/72 \\ -1/60 & 4/45 & -1/12 \\ 1/72 & -1/12 & 1/12 \end{bmatrix}$
- $\det \mathbf{A} = \frac{1}{240} - \frac{1}{120} + \frac{1}{216} = \frac{1}{2160}$
- Verify $\mathbf{C}^\top \mathbf{A} = (\det \mathbf{A})\mathbf{I}$
- $\mathbf{A}^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$