MA 106 : Linear Algebra

Tutorial 2

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17^{th} March 2021

Question 1

Find the Row Canonical Form of $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 \to -R_3, R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 - 2R_2}$$

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 - 3R_3, R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Question 2

Let $\mathbf{A} := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Find $\mathbf{A}^{-1}$ by Gauss-Jordan method.

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1, R_2 - R1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Thus $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Elementary Matrices

An $m \times m$ matrix $\mathbf{E}$ is called an **elementary matrix** if it is obtained from the identity matrix $\mathbf{I}$ by an elementary row operation.

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ So } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is the elem. matrix corr. to ERO } R_1 \leftrightarrow R_2.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ So } \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is the elem. matrix corr. to ERO } R_1 + 2R_2.$$

Question 3 (i)

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. If an elementary row operation transforms $\mathbf{A}$ to $\mathbf{A}'$, then show that $\mathbf{A}' = \mathbf{EA}$, where $\mathbf{E}$ is the corresponding elementary matrix.

Example

The ERO $R_2 + 2R_1$ transforms $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ to $\mathbf{A}' = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$

The corresponding elementary matrix can be obtained by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ *i.e.*

$$\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Now observe that $\mathbf{EA} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix} = \mathbf{A}'$

Question 3(i) Contd ..

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. If an elementary row operation transforms $\mathbf{A}$ to $\mathbf{A}'$, then show that $\mathbf{A}' = \mathbf{EA}$, where $\mathbf{E}$ is the corresponding elementary matrix.

Hint: $$\mathbf{I} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_m \end{bmatrix} \xrightarrow{R_1 + \alpha R_j} \begin{bmatrix} \mathbf{e}_1 + \alpha \mathbf{e}_j \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_m \end{bmatrix} = \mathbf{E};\ \mathbf{EA} = \begin{bmatrix} a_{11} + \alpha a_{j1} & \cdots & a_{1n} + \alpha a_{jn} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Can you prove it in general for all 3 EROs?

Question 3 (ii)

Show that every elementary matrix is invertible, and find its inverse.

- An square matrix is invertible if and only if it can be transformed to the identity matrix by EROs (Shown in class)
- Since an elementary matrix is obtained by a single ERO on the identity matrix, we can get back the identity matrix by an ERO on the elementary matrix.
- Why should such an ERO exist?
  - $R_i \leftrightarrow R_j$ can be reversed by $R_i \leftrightarrow R_j$
  - $R_i + \alpha R_j$ by $R_i - \alpha R_j$
  - αR_i by $\frac{1}{\alpha} R_i$

For $i \neq j$,

- For the elementary matrix $\mathbf{E}_1$ corr. to $R_i \leftrightarrow R_j$, $\mathbf{E}_1^{-1} = \mathbf{E}_1$
- For $\mathbf{E}_2$ corr. to $R_i + \alpha R_j$, $\mathbf{E}_2^{-1} = \mathbf{E}_3$, where $\mathbf{E}_3$ is the elem. matrix corr. to $R_i - \alpha R_j$
- For $\mathbf{E}_4$ corr. to ERO αR_i, $\mathbf{E}_4^{-1} = \mathbf{E}_5$, where $\mathbf{E}_5$ is the elem. matrix corr. to $\frac{1}{\alpha} R_i$

Question 3 (ii) Contd ..

Example

- For the elementary matrix $\mathbf{E}_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the ERO $R_1 - 2R_2$ gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- The elementary matrix corresponding to the ERO $R_1 - 2R_2$ is $\mathbf{E}_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Thus we have $\mathbf{E}_2\mathbf{E}_1 = \mathbf{I}$ (From Question 3 (i))
- For $\mathbf{A} \in \mathbb{R}^{n \times n}$, if there is $\mathbf{B} \in \mathbb{R}^{n \times n}$ such that either $\mathbf{BA} = \mathbf{I}$ or $\mathbf{AB} = \mathbf{I}$, then $\mathbf{A}$ is invertible, and $\mathbf{A}^{-1} = \mathbf{B}$ (Shown in class)
- From the above two statement, we have that $\mathbf{E}_1^{-1} = \mathbf{E}_2$

Question 3 (iii)

(a) If a square matrix $\mathbf{A}$ is a product of elementary matrices, it is invertible

- If square matrices $\mathbf{A}$ and $\mathbf{B}$ are invertible, so is $\mathbf{AB}$ (Proved in class)
- $\mathbf{A} = \mathbf{E}_1\mathbf{E}_2 \dots \mathbf{E}_k$ and each $\mathbf{E}_i$ is invertible
- Can you complete the proof?

(b) If a square $\mathbf{A}$ is invertible, it is a product of elementary matrices

- Every square invertible matrix can be converted to $\mathbf{I}$ by EROs (Proved in class)
- We have shown that EROs are equivalent to multiplication by corr. elementary matrices
- Thus, there exists $\mathbf{E}_1, \mathbf{E}_2, \dots \mathbf{E}_k$ such that $\mathbf{E}_k\mathbf{E}_{k-1} \dots \mathbf{E}_1\mathbf{A} = \mathbf{I}$
- Or $\mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1} \dots \mathbf{E}_k^{-1}$
- We have also shown that inverse of an elementary matrix is an elementary matrix
- So $\mathbf{A}$ is a product of elementary matrices.

Techniques for proving $A \to B$

- **Direct Proof**: Assume A and through a series of steps, arrive at B
- **Proof by Contrapositive**: Assume $\neg B$, arrive at $\neg A$. This works since $A \to B \equiv \neg A \vee B \ \equiv \ \neg(\neg B) \vee \neg A \ \equiv \ \neg B \to \neg A$
- **Proof by Contradiction**: Assume A and $\neg B$ and derive a contradiction [1]. This works since, it can be shown that $A \to B \equiv (A \wedge \neg B) \to \mathrm{FALSE}$
- We will be using proof by contradiction in the following question

[1] For a more general form, see Wiki:Proof by Contradiction

Question 4

Let S and T be subsets of $\mathbb{R}^{n\times 1}$ such that $S \subset T$. Show that if S is linearly dependent then so is T. If T is linearly independent then so is S.

- Let $S = \{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_s\}$ and $T = S \cup \{\mathbf{a}_{s+1}, \mathbf{a}_{s+2} \ldots, \mathbf{a}_{s+t}\}$, where $s = |S|$
- Since $S \subset T$, $|T \setminus S| > 0$ *i.e* $t > 0$
- Given S is linearly dependent,
  - We have $\alpha_1 \ldots \alpha_s$, not all zero, such that $\sum_{j=1}^{s} \alpha_j \mathbf{a}_j = \mathbf{0}$
  - Define $\{\alpha_{s+1}, \alpha_{s+2} \ldots \alpha_{s+t}\}$, all equal to 0
  - $\sum_{j=1}^{s} \alpha_j \mathbf{a}_j + \sum_{j=1}^{t} \alpha_{s+j} \cdot \mathbf{a}_{s+j} = \mathbf{0} \implies \sum_{j=1}^{s+t} \alpha_j \mathbf{a}_j = \mathbf{0}$
  - These are exactly the set of vectors in T. Thus T is linearly dependent.
- For contradiction, assume T is linearly independent and S is not
  - Since S is not linearly independent, $\exists\ \alpha_j$, not all zero, s.t. $\sum_{j=1}^{s} \alpha_j \mathbf{a}_j = \mathbf{0}$
  - Choosing $\{\alpha_{s+1}, \alpha_{s+2} \ldots \alpha_{s+t}\}$, all equal to 0, we get $\sum_{j=1}^{s+t} \alpha_j \mathbf{a}_j = \mathbf{0}$
  - $\Rightarrow$ T is linearly dependent, a contradiction (to our assumption that T is linearly independent)

Question 4 Contd ..

Converse

(i) T is linearly dependent $\Rightarrow$ S is also linearly dependent

$$T = \{[1,0],[1,1],[0,1]\}, S = \{[1,0],[0,1]\}$$

(ii) S is linearly independent $\Rightarrow$ T is also linearly independent

$$S = \{[1,0],[0,1]\}, T = \{[1,0],[1,1],[0,1]\}$$

Question 5

Are the following sets linearly independent?

(i) $\left\{\begin{bmatrix}1 & -1 & 1\end{bmatrix}, \begin{bmatrix}3 & 5 & 2\end{bmatrix}, \begin{bmatrix}1 & 2 & 1\end{bmatrix}, \begin{bmatrix}1 & 1 & 1\end{bmatrix}\right\} \subset \mathbb{R}^{1\times 3}$

(ii) $\left\{\begin{bmatrix}1 & 9 & 9 & 8\end{bmatrix}, \begin{bmatrix}2 & 0 & 0 & 3\end{bmatrix}, \begin{bmatrix}2 & 0 & 0 & 8\end{bmatrix}\right\} \subset \mathbb{R}^{1\times 4}$

(iii) $\left\{\begin{bmatrix}1 & -1 & 0\end{bmatrix}^\top, \begin{bmatrix}3 & -5 & 2\end{bmatrix}^\top, \begin{bmatrix}1 & -2 & 1\end{bmatrix}^\top\right\} \subset \mathbb{R}^{3\times 1}$

(i) Follows from the fact that if S is a set of vectors of length n and if S has more than n elements, then S is linearly dependent.

(ii) $\begin{bmatrix}1 & 9 & 9 & 8\\ 2 & 0 & 0 & 3\\ 2 & 0 & 0 & 8\end{bmatrix} \to \begin{bmatrix}1 & 9 & 9 & 8\\ 2 & 0 & 0 & 3\\ 0 & 0 & 0 & 5\end{bmatrix} \to \begin{bmatrix}1 & 9 & 9 & 8\\ 0 & -18 & -18 & -13\\ 0 & 0 & 0 & 5\end{bmatrix}$

3 non-zero rows, so row rank is 3 or the 3 rows are linearly independent

(iii) Show REF of $\begin{bmatrix}1 & 3 & 1\\ -1 & -5 & -2\\ 0 & 2 & 1\end{bmatrix}$ has 3 non-zero rows

Question 6

Given a set of s linearly independent row vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_i, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_s\}$ in $\mathbb{R}^{1\times n}$ and $\alpha \in \mathbb{R}$, show that the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_i + \alpha\mathbf{a}_j, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_s\}$ is linearly independent.

- For the sake of contradiction, suppose that $\{\mathbf{a}_i\}$ are linearly independent but $\{\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_i + \alpha\mathbf{a}_j, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_s\}$ are not linearly independent
- Thus these exists scalars $c_1, c_2 \ldots c_s$, not all 0, such that

$$\sum_{k=1}^{i-1} c_k\mathbf{a}_k + c_i(\mathbf{a}_i + \alpha\mathbf{a}_j) + c_j\mathbf{a}_j + \sum_{k=i+1,k\neq j}^{s} c_k\mathbf{a}_k = 0$$

$$\sum_{k=1}^{i-1} c_k\mathbf{a}_k + c_i\mathbf{a}_i + (c_i\alpha + c_j)\mathbf{a}_j + \sum_{k=i+1,k\neq j}^{s} c_k\mathbf{a}_k = 0$$

- So there exist $c'_1, c'_2 \ldots c'_s$, not all zero, such that $\sum c'_k\mathbf{a}_k = 0$
- What are $c'_1 \ldots c'_s$ in terms of $c_1, \ldots c_s$? Why is there a non-zero c'_k?
- This contradicts our assumption that $\{\mathbf{a}_i\}$ are linearly independent

Question 7 (i)

Find the rank of the given matrix

- We know that elementary row operations do not change rank
- $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix} \xrightarrow{R_2 \to -R_2, R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & -1 \\ 8 & -4 \\ 6 & -3 \end{bmatrix} \xrightarrow{?} \begin{bmatrix} 2 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
- One non-zero row (one pivot), so rank=1
- $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$ (rank $= 3$)