MA 106 : Linear Algebra

Tutorial 1

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Logistics

- Tutorial: Wednesdays 2-3 PM on MS Teams
- Use Discussion Forum on Teams for posting doubts
- Contact via email (170070010@iitb.ac.in) or using MS Teams chat
- Some steps and/or reasons might be omitted in the slides
- In case of difference in notation, please follow the ones used in class for the exams
- Question?

Question 1

Let $\mathbf{A}$ be a square matrix. Show that there is a symmetric matrix $\mathbf{B}$ and there is a skew-symmetric matrix $\mathbf{C}$ such that $\mathbf{A} = \mathbf{B} + \mathbf{C}$. Are $\mathbf{B}$ and $\mathbf{C}$ unique?

- Given $\mathbf{A} = \mathbf{B} + \mathbf{C}$
- We have $\mathbf{A}^\top = \mathbf{B}^\top + \mathbf{C}^\top$
- Now since $\mathbf{B}$ is symmetric and $\mathbf{C}$ is skew-symmetric, we have

$$\mathbf{B}^\top = \mathbf{B} \quad \text{and} \quad \mathbf{C}^\top = -\mathbf{C}$$

- This gives us $\mathbf{A}^\top = \mathbf{B} - \mathbf{C}$
- Using $\mathbf{A} = \mathbf{B} + \mathbf{C}$ and $\mathbf{A}^\top = \mathbf{B} - \mathbf{C}$, we have

$$\mathbf{B} = \frac{\mathbf{A} + \mathbf{A}^\top}{2} \quad \& \quad \mathbf{C} = \frac{\mathbf{A} - \mathbf{A}^\top}{2}$$

- $\mathbf{B}$ and $\mathbf{C}$ are unique.

Recall

- The jth row of **AB** is a linear combination of the row vectors of **B** with coefficients $a_{j1}, a_{j2} \dots$ provided by the jth row of **A**.
- The kth column of **AB** is a linear combination of the column vectors of **A** with coefficients $b_{1k}, b_{2k} \dots$ provided by the kth column of **B**.

Question 2

Let $\mathbf{A} := \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{B} := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Write
(i) the second row of **AB** as a linear combination of rows of **B**
(ii) the second column of **AB** as a linear combination of columns of **A**

- Let $\mathbf{C} = \mathbf{AB}$. Then $c_{ij} = \mathbf{a}_i \mathbf{b}_j$ where $\mathbf{a}_i$ is the i^{th} row of **A** and $\mathbf{b}_j$ is the j^{th} column of **B**
- Second row of $\mathbf{C} = \mathbf{a}_2\mathbf{B} = [\mathbf{a}_2\mathbf{b}_1,\ \mathbf{a}_2\mathbf{b}_2,\ \mathbf{a}_2\mathbf{b}_3]$
- $[\mathbf{a}_2\mathbf{b}_1,\ \mathbf{a}_2\mathbf{b}_2,\ \mathbf{a}_2\mathbf{b}_3] = [a_{21}b_{11}+a_{22}b_{21},\ a_{21}b_{12}+a_{22}b_{22},\ a_{21}b_{13}+a_{22}b_{23}]$
  $= [a_{21}b_{11},\ a_{21}b_{12},\ a_{21}b_{13}] + [a_{22}b_{21},\ a_{22}b_{22},\ a_{22}b_{23}] = a_{21}\mathbf{b}'_1 + a_{22}\mathbf{b}'_2$
- $\mathbf{b}'_1 = [b_{11},\ b_{12},\ b_{13}]$ and $\mathbf{b}'_2 = [b_{21},\ b_{22},\ b_{23}]$ are the rows of **B**
- (i) $3\ [1\ 2\ 3] + 4\ [4\ 5\ 6]$
- (ii) $\mathbf{Ab}_2 = 2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

Question 3

Let $\mathbf{A} := \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -24 & 8 & -5 \\ 0 & -7 & 2 & 2 \end{bmatrix}$.

Assuming that $\mathbf{A}$ is invertible, find the last column and the last row of $\mathbf{A}^{-1}$

- Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top$ be the last column of $\mathbf{A}^{-1}$
- We know that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$
- We also know that the k^{th} column of $\mathbf{AB}$ is given by $\sum_i b_{ik}\mathbf{a}_i$ where $\mathbf{a}_i$ is the i^{th} column of $\mathbf{A}$
- Thus the fourth column of $\mathbf{A}\mathbf{A}^{-1}$ is $\left(\mathbf{e}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top\right)$

$$x_1[1 \ -3 \ 4 \ 0]^\top + x_2[1 \ -17 \ -24 \ -7]^\top + x_3[\cdots]^\top + x_4[\cdots]^\top = \mathbf{e}_4$$

- 4 equations in 4 variables. Solve for $\mathbf{x}$ (Reduce to row echelon form?)
- For the last row of $\mathbf{A}^{-1}$, use $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ to get
- $y_1[1 \ 1 \ 1 \ 0] + y_2[-3 \ -17 \ 1 \ 2] + y_3[\cdots] + y_4[0 \ -7 \ 2 \ 2] = \mathbf{e}_4^\top$

Question 4

Show that the product of two upper triangular matrices is upper triangular. Is this true for lower triangular matrices?

- Let $\mathbf{A} \in \mathbb{R}^{l,m}$ and $\mathbf{B} \in \mathbb{R}^{m,n}$ be two upper triangular matrices
- By the definition of upper triangular matrices, $a_{ij} = b_{ij} = 0 \ \forall i > j$
- Let $\mathbf{C} = \mathbf{AB}$
- We know that $c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^{m} a_{ik} b_{kj}$

$$c_{ij} = \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^{m} a_{ik} b_{kj}$$

- For $i > j$,
  - In the first summation, $i > k$ so $a_{ik} = 0$
  - In the second summation, $k > j$ so $b_{kj} = 0$
- $\therefore c_{ij} = 0 \ \forall i > j$. Thus $\mathbf{C} = \mathbf{AB}$ is upper triangular
- If $\mathbf{A}$ is lower triangular, then $\mathbf{A}^\top$ is upper triangular and vice versa. Prove this and use it to answer the second part.

Question 5

The trace of a square matrix is the sum of its diagonal entries. Show that $\text{trace}(\mathbf{A}+\mathbf{B}) = \text{trace}(\mathbf{A}) + \text{trace}(\mathbf{B})$ and $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$ for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.

- $\text{trace}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$
- $\text{trace}(\mathbf{A}+\mathbf{B}) = \sum_{i=1}^{n}(a_{ii}+b_{ii})$ (Why?)
  $= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$
  $= \text{trace}(\mathbf{A}) + \text{trace}(\mathbf{B})$
- Let $\mathbf{C} = \mathbf{AB}$ and $\mathbf{D} = \mathbf{BA}$
- $\text{trace}(\mathbf{AB}) = \sum_{i=1}^{n} c_{ii}$
  $= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}$
  $= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij}$
  $= \sum_{j=1}^{n} d_{jj}$
  $= \text{trace}(\mathbf{BA})$
- Write reasons for each (non-trivial) step in exams

Question 6(i)

Find all solutions of $\mathbf{Ax} = \mathbf{b}$ by reducing $\mathbf{A}$ to a row echelon form.

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$$

$R_2 \to R_2 - 2R_1,\ R_4 \to R_4 - 2R_1$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 6 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right]$$

$R_3 \to R_3 + 5R_2, R_4 \to R_4 + 4R_2$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right]$$

Question 6(i) Contd.

Find all solutions of $\mathbf{Ax} = \mathbf{b}$ by reducing $\mathbf{A}$ to a row echelon form.

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right]$$

$R_3 \leftrightarrow R_4$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The last row gives $0 = 1$. Thus no solution exists.

Question 6 (ii)

Find all solutions of $\mathbf{Ax} = \mathbf{b}$ by reducing $\mathbf{A}$ to a row echelon form.

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 + R_3$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$2x_1 + x_2 + x_3 = 5$, $-8x_2 - 2x_3 = -12$ and $x_3 = 2$

By back substitution, $x_3 = 2$, $x_2 = 1$, $x_1 = 1$

$$\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^\top$$

Question 6 (iii)

Find all solutions of $\mathbf{Ax} = \mathbf{b}$ by reducing $\mathbf{A}$ to a row echelon form.

$$\left[\begin{array}{cccc|c} 0 & 2 & -2 & 1 & 2 \\ 2 & -8 & 14 & -5 & 2 \\ 1 & 3 & 0 & 1 & 8 \end{array}\right]$$

$R_3 \leftrightarrow R_1$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 2 & -8 & 14 & -5 & 2 \\ 0 & 2 & -2 & 1 & 2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 0 & -14 & 14 & -7 & -14 \\ 0 & 2 & -2 & 1 & 2 \end{array}\right]$$

$R_3 \to R_3 + \frac{1}{7}R_2$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 0 & -14 & 14 & -7 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Question 6 (iii) Contd..

$R_2 \to -\frac{1}{7}R_2$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 8 \\ 0 & 2 & -2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Particular Solution: Set $x_3 = x_4 = 0$. By back substitution, we get $x_2 = 1, x_1 = 5$ *i.e* $\mathbf{x}_0 = \begin{bmatrix} 5 & 1 & 0 & 0 \end{bmatrix}^\top$
- Basic solutions:
  - $x_3 = 1, x_4 = 0$ gives $\mathbf{s}_1 = \begin{bmatrix} -3 & 1 & 1 & 0 \end{bmatrix}^\top$
  - $x_4 = 1, x_3 = 0$ gives $\mathbf{s}_2 = \begin{bmatrix} -5/2 & 1/2 & 0 & 1 \end{bmatrix}^\top$
- All Solutions (*i.e.* General solution) : $\mathbf{x} = \mathbf{x}_0 + \alpha_1\mathbf{s}_1 + \alpha_2\mathbf{s}_2$ where $\alpha_1, \alpha_2 \in \mathbb{R}$